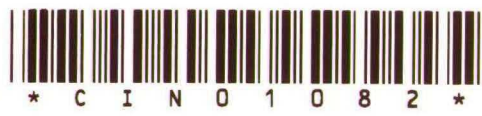


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**ASYMPTOTIC ANALYSIS OF NASH EQUILIBRIA IN NONZERO-SUM LINEAR-QUADRATIC DIFFERENTIAL GAMES. THE TWO-PLAYER CASE.**

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J.C. Engwerda

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# ASYMPTOTIC ANALYSIS OF NASH EQUILIBRIA IN NONZERO-SUM LINEAR-QUADRATIC DIFFERENTIAL GAMES. THE TWO-PLAYER CASE.

A.J.T.M. WEEREN, J.M. SCHUMACHER, AND J.C. ENGWERDA

**ABSTRACT.** In this paper we discuss nonzero-sum linear-quadratic differential games. Already in the papers by Starr and Ho [13, 14], for this kind of games the Nash equilibria for different kinds of information structures were studied. Most of the literature on the topic of nonzero-sum linear-quadratic differential games is concerned with games of fixed (finite) duration, i.e. the games are studied over a finite time horizon  $t_f$ . We will study the behaviour of Nash equilibria for  $t_f \rightarrow \infty$  for two different information structures, the open-loop and the closed-loop perfect state information.

In the open-loop case, it is known from [1, 2] that the coupled Riccati differential equations describing the Nash equilibrium can be related to a linear dynamical system. Using this linear system, asymptotic properties of the open-loop Nash equilibrium are studied.

In the case of closed-loop perfect state information, we will study the so-called feedback Nash equilibrium. The equations for the feedback Nash equilibrium are inherently nonlinear and we will limit the dynamic analysis to the scalar case. For the special case that all parameters are scalar, a detailed dynamical analysis is given for the quadratic system of coupled Riccati equations. We show that there exist choices of parameters, for which the asymptotic behaviour of the solutions of the Riccati equations depends strongly on the specified terminal values. Finally, we show that, although the feedback Nash equilibrium over any fixed finite horizon is unique, there can exist several different feedback Nash equilibria in stationary strategies for the infinite horizon problem, even when we restrict our attention to Nash equilibria that are stable in the dynamical sense of the word.

## 1. INTRODUCTION

Differential games were first introduced by Isaacs [7], within the framework of two-person zero-sum games. Recently, the theory of zero-sum differential games has successfully been

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used in the area of  $H_\infty$  control theory, see e.g. [4]. Nonzero-sum differential games were introduced in the papers by Starr and Ho [13, 14]. A good survey of the area of dynamic games is provided in the book by Başar and Olsder [3].

In this paper we look at a special class of nonzero-sum differential games, namely nonzero-sum differential games of the linear-quadratic type. The dynamics are supposed to be described by a linear differential equation,

$$(1) \quad \dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \quad x(0) = x_0,$$

and for each player a quadratic cost functional is given:

$$(2) \quad \begin{aligned} L_1(u_1, u_2) &:= x(t_f)' K_{1f} x(t_f) + \\ &\int_0^{t_f} \{x(t)' Q_1 x(t) + u_1(t)' R_{11} u_1(t) + u_2(t)' R_{12} u_2(t)\} dt, \\ L_2(u_1, u_2) &:= x(t_f)' K_{2f} x(t_f) + \\ (3) \quad &\int_0^{t_f} \{x(t)' Q_2 x(t) + u_1(t)' R_{21} u_1(t) + u_2(t)' R_{22} u_2(t)\} dt, \end{aligned}$$

in which all matrices are symmetric, and moreover  $Q_i \geq 0$  and  $R_{ii} > 0$ .

The objective of the game for each player is the minimization of his own cost functional by choosing appropriate inputs for the underlying linear dynamical system.

For given information sets  $\eta_i(t)$  and any pair of strategies  $(\gamma_1, \gamma_2)$ , the actions of the players are completely determined by the relations  $(u_1, u_2) = (\gamma_1(\eta_1), \gamma_2(\eta_2))$ . Substitution of the pair  $(u_1, u_2)$  in (2–3), together with the corresponding unique state trajectory, yields a pair of numbers  $(L_1(u_1, u_2), L_2(u_1, u_2))$ . Therefore we have a mapping for each fixed initial state vector  $x_0$ , defined by

$$(4) \quad J_i : \Gamma_1 \times \Gamma_2 \rightarrow \mathbb{R}, (\gamma_1, \gamma_2) \mapsto L_i(u_1, u_2),$$

which we call the cost functional of player  $i$  for the game in normal form.

In [8] the Nash equilibrium concept was introduced, which was argued to be a natural concept in a noncooperative context. The Nash equilibrium is defined in the following way (see e.g. [3]):

**Definition 1.1.** A pair of strategies  $(\gamma_1^*, \gamma_2^*) \in \Gamma_1 \times \Gamma_2$  is a Nash equilibrium for the differential game, if for all  $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$  the following inequalities hold:

$$(5) \quad J_1(\gamma_1^*, \gamma_2^*) \leq J_1(\gamma_1, \gamma_2^*),$$

$$(6) \quad J_2(\gamma_1^*, \gamma_2^*) \leq J_2(\gamma_1^*, \gamma_2).$$

The Nash equilibrium is defined such that it has the property that there is no incentive for any unilateral deviation by any one of the players. A possible problem when dealing with Nash equilibria, is that in general one cannot expect to have a unique Nash equilibrium. Already in



the paper [14], for nonzero-sum differential games, non-uniqueness problems regarding Nash equilibria were discussed, related to different information structures for the game.

In almost all papers on linear-quadratic differential games, the games are studied over a fixed time period  $[0, t_f]$ . In the case of open-loop information, where every player knows at time  $t \in [0, t_f]$  the initial state  $x_0$  (denoted by  $\eta_i(t) = \{x_0\}$ ), conditions for the existence of a unique Nash equilibrium can be given. In the case of closed-loop perfect state information, where every player knows at time  $t \in [0, t_f]$  the complete history of the state (denoted by  $\eta_i(t) = \{x(s) \mid 0 \leq s \leq t\}$ ), one can show that there exist many Nash equilibria. In this case it is possible to define a refinement of the Nash equilibrium concept towards feedback Nash equilibria, which have the nice property of strong time consistency. In the finite (fixed) horizon case, uniqueness of the feedback Nash equilibrium can be shown (see e.g. [3]).

Only a few authors have studied the game over an infinite time horizon, or the asymptotic behaviour of Nash equilibria for  $t_f \rightarrow \infty$ . In the paper by Abou-Kandil et al. [2], the asymptotic behaviour of open-loop Nash equilibria is studied. For the feedback Nash equilibrium, in the paper by Papavassilopoulos [9], an initial study is made of infinite horizon feedback Nash equilibria. However, in the paper [9] the asymptotic behaviour of finite horizon feedback Nash equilibria is not studied. Also, the problem of existence of infinite horizon feedback Nash equilibria is not addressed. Instead, some sufficient solvability conditions for the coupled algebraic Riccati equations are derived, using Brouwer's fixed point theorem.

In section 2 of this paper, we discuss the asymptotic analysis of Nash equilibria in the open-loop case, based on the fact that in the open-loop case the related coupled Riccati equations can be related to a linear differential system (see papers [1, 2]). In section 3, we will show that for feedback Nash equilibria it is not possible to follow a similar approach. Instead we will give a detailed asymptotic analysis for the special case that all system parameters are scalar. Finally, in section 4, we study infinite horizon feedback Nash equilibria, and use the results from section 3 to show that in the infinite horizon case we no longer have uniqueness of feedback Nash equilibria.

## 2. OPEN-LOOP NASH EQUILIBRIA

In this section we study the open-loop information structure, i.e.  $\eta_i(t) = \{x_0\}$ ,  $t \in [0, t_f]$ . The following theorem is well-known (see [3]):

**Theorem 2.1.** *The pair of strategies, given by*

$$(7) \quad u_1(t) = \gamma_1(t, x_0) = -R_{11}^{-1} B_1' K_1(t) \Psi(t, 0) x_0,$$

$$(8) \quad u_2(t) = \gamma_2(t, x_0) = -R_{22}^{-1} B_2' K_2(t) \Psi(t, 0) x_0,$$

is a Nash equilibrium for the open-loop information structure. Here  $K_1(t)$  and  $K_2(t)$  are given by the asymmetric coupled Riccati differential equations

$$(9) \quad \dot{K}_1 = -A'K_1 - K_1A - Q_1 + K_1S_1K_1 + K_1S_2K_2,$$

$$(10) \quad \dot{K}_2 = -A'K_2 - K_2A - Q_2 + K_2S_2K_2 + K_2S_1K_1,$$

$$(11) \quad K_1(t_f) = K_{1f},$$

$$(12) \quad K_2(t_f) = K_{2f}.$$

in which  $S_1 = B_1R_{11}^{-1}B_1'$  and  $S_2 = B_2R_{22}^{-1}B_2'$ . Furthermore  $\Psi(t, s)$  is given by

$$(13) \quad \dot{\Psi}(t) = (A - S_1K_1(t) - S_2K_2(t))\Psi(t, s), \quad \Psi(t, t) = I$$

Given the fixed time horizon  $t_f$ , we have an expression for the open-loop Nash equilibrium, stated in terms of solutions to the coupled Riccati equations (9–12). These equations can be related to a linear differential system (see [1, 2]). Define the matrix  $M$  in the following way:

$$(14) \quad M := \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A' & 0 \\ Q_2 & 0 & A' \end{pmatrix}.$$

Then we can characterize the solution of the differential equations (9–12) by means of

$$(15) \quad e^{M\tau} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} = \text{span} \begin{pmatrix} I \\ K_1(\tau) \\ K_2(\tau) \end{pmatrix},$$

where  $\tau = t_f - t, t \in [0, t_f]$ .

The characterization (15) not only allows for explicit calculation of the solutions of the coupled differential equations (9–12) in a given specific case, but it also facilitates the asymptotic analysis of the open-loop Nash equilibrium (see also [1, 2]). Before we can give the main theorem from [2], we first need the following definition:

**Definition 2.2.** Let  $M \in \mathbb{R}^{3n \times 3n}$ .

- (i)  $M$  is called *dichotomically separable* if there exist subspaces  $X_1$  and  $X_2$  of  $\mathbb{R}^{3n}$ , such that  $MX_1 \subset X_1$ ,  $MX_2 \subset X_2$ ,  $X_1 \oplus X_2 = \mathbb{R}^{3n}$ ,  $\dim X_1 = n$ , and  $\text{Re}\sigma(M|_{X_1}) > \text{Re}\sigma(M|_{X_2})$ .
- (ii)  $M$  is called *reverse dichotomically separable* if there exist subspaces  $X_1$  and  $X_2$  of  $\mathbb{R}^{3n}$ , such that  $MX_1 \subset X_1$ ,  $MX_2 \subset X_2$ ,  $X_1 \oplus X_2 = \mathbb{R}^{3n}$ ,  $\dim X_1 = n$ , and  $\text{Re}\sigma(M|_{X_1}) < \text{Re}\sigma(M|_{X_2})$ .

Then the main theorem of [2] can be formulated in the following way:

**Theorem 2.3.** *Let  $M$  be the matrix given by (14).*

- (i) *If  $M$  is dichotomically separable, there exists a constant solution  $(K_1^\omega, K_2^\omega)$  of the coupled differential equations (9-12), which has an open and dense domain of attraction for  $t_f \rightarrow \infty$ .*
- (ii) *If  $M$  is reverse dichotomically separable, then there exists a constant solution  $(K_1^\alpha, K_2^\alpha)$  of the coupled differential equations (9-12), which has an open and dense domain of attraction for  $t_f \rightarrow -\infty$ .*
- (iii)  *$(K_1^\omega, K_2^\omega)$  (or  $(K_1^\alpha, K_2^\alpha)$ ) are the only stable equilibrium solutions of (9-12) for  $t_f \rightarrow \infty$  (or  $t_f \rightarrow -\infty$ ).*

We conclude this section on open-loop Nash equilibria with an example. In this example we show that the theorem 2.3 does not tell the complete story. We show in the simplest possible case (dimension of the state space  $n = 2$ ), that there exist matrices  $M$  of the form (14), that are neither dichotomically separable nor reverse dichotomically separable, but where the coupled differential equations (9-12) do allow for constant solutions.

**Example 2.4.** In this example, we choose the following matrices:

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$B_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$B_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$Q_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Q_2 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then  $M$  is given by

$$M := \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

with eigenvalues

$$\sigma M = \left\{ -\sqrt{2}, -\frac{1}{2}\sqrt{2}\sqrt{1+\sqrt{2}} \pm \frac{1}{2}i\sqrt{2}\sqrt{1+\sqrt{2}}, \frac{1}{2}\sqrt{2}\sqrt{1+\sqrt{2}} \pm \frac{1}{2}i\sqrt{2}\sqrt{1+\sqrt{2}}, \sqrt{2} \right\}.$$

Hence  $M$  is neither dichotomically separable nor reverse dichotomically separable. Yet it is easy to show that there exist exactly two constant solutions to the coupled differential equations (9–12), namely  $(K_1^o, K_2^{(1)})$  and  $(K_1^o, K_2^{(2)})$ , where

$$\begin{aligned} K_1^o &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \\ K_2^{(1)} &= \begin{pmatrix} 2\sqrt{1+\sqrt{2}} & 1+\sqrt{2} \\ 1+\sqrt{2} & \sqrt{2}\sqrt{1+\sqrt{2}} \end{pmatrix}, \\ K_2^{(2)} &= \begin{pmatrix} -2\sqrt{1+\sqrt{2}} & 1+\sqrt{2} \\ 1+\sqrt{2} & -\sqrt{2}\sqrt{1+\sqrt{2}} \end{pmatrix}. \end{aligned}$$

Furthermore it is easy to show that both equilibria are neither stable nor unstable. This implies in particular that, for generic terminal conditions  $(K_{1f}, K_{2f})$ , there will be no convergence of the corresponding solution of the open-loop coupled Riccati differential equations (9–12) to a constant solution.

### 3. FEEDBACK NASH EQUILIBRIA

**3.1. Introduction.** In this section we study closed-loop perfect state (CLPS) information, i.e.  $\eta_i(t) = \{x(s) \mid 0 \leq s \leq t\}$ ,  $t \in [0, t_f]$ . For this information structure the following theorem is well-known (see [3, 13]):

**Theorem 3.1.** *Let the strategies  $(\gamma_1^*, \gamma_2^*)$  be such that there exist solutions  $(\psi_1, \psi_2)$  to the differential equations*

$$\begin{aligned} (16) \quad \dot{\psi}_1 &= -\frac{\partial H_1}{\partial x}(x^*, \gamma_1^*(x^*), \gamma_2^*(x^*), \psi_1) \\ &\quad - \frac{\partial H_1}{\partial u_2}(x^*, \gamma_1^*(x^*), \gamma_2^*(x^*), \psi_1) \cdot \frac{\partial \gamma_2^*}{\partial x}(x^*) \end{aligned}$$

$$\begin{aligned} (17) \quad \dot{\psi}_2 &= -\frac{\partial H_2}{\partial x}(x^*, \gamma_1^*(x^*), \gamma_2^*(x^*), \psi_2) \\ &\quad - \frac{\partial H_2}{\partial u_1}(x^*, \gamma_1^*(x^*), \gamma_2^*(x^*), \psi_2) \cdot \frac{\partial \gamma_1^*}{\partial x}(x^*) \end{aligned}$$

in which, for  $i = 1, 2$ ,

$$(18) \quad H_i(x, u_1, u_2, \psi_i) := x'Q_i x + u_1'R_{i1}u_1 + u_2'R_{i2}u_2 + \psi_i'(Ax + B_1u_1 + B_2u_2)$$



with terminal conditions, for  $i = 1, 2$ ,

$$(19) \quad \psi_i(t_f) = K_{if} x^*(t_f)$$

such that for  $i = 1, 2$ ,

$$(20) \quad \frac{\partial H_i}{\partial u_i}(x^*, \gamma_1^*(x^*), \gamma_2^*(x^*), \psi_i) = 0$$

and  $x^*$  satisfies

$$(21) \quad \dot{x}^*(t) = Ax^*(t) + B_1 \gamma_1^*(x^*(t), t) + B_2 \gamma_2^*(x^*(t), t),$$

$$(22) \quad x^*(0) = x_0.$$

Then  $(\gamma_1^*, \gamma_2^*)$  is a Nash equilibrium with respect to the CLPS information structure, and is given by  $u_i^*(t) = \gamma_i^*(x^*, t) = -R_{ii}^{-1} B_i' \psi_i(t)$ .

*Remark.* Let  $(\gamma_1^{ol}, \gamma_2^{ol})$  be an open-loop Nash equilibrium. Then it is easily seen that  $(\gamma_1^{ol}, \gamma_2^{ol})$  is also a Nash equilibrium with respect to the CLPS information structure, because  $\frac{\partial}{\partial x} \gamma_i^{ol} = 0$ .

When we restrict the admissible strategies to the class of (possibly timevarying) linear feedback strategies, i.e.  $\Gamma_i^{fb} := \{\gamma_i \mid \gamma_i(x, t) = F_i(t)x\}$ , then there exists a unique feedback Nash equilibrium (see e.g. [3, section 6.5.1.6.5.2]). The following theorem can be found in [3, 13].

**Theorem 3.2.** Suppose  $(K_1, K_2)$  satisfy the coupled Riccati equations, given by

$$(23) \quad \dot{K}_1 = -A'K_1 - K_1A - Q_1 + K_1S_1K_1 + K_1S_2K_2 + K_2S_2K_1 - K_2S_{02}K_2$$

$$(24) \quad \dot{K}_2 = -A'K_2 - K_2A - Q_2 + K_2S_2K_2 + K_2S_1K_1 + K_1S_1K_2 - K_1S_{01}K_1$$

$$(25) \quad K_1(t_f) = K_{1f}$$

$$(26) \quad K_2(t_f) = K_{2f}$$

where  $S_1 = B_1 R_{11}^{-1} B_1'$ ,  $S_2 = B_2 R_{22}^{-1} B_2'$ ,  $S_{01} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1'$  and  $S_{02} = B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2'$ .

Then the pair of strategies  $(\gamma_1^*(x, t), \gamma_2^*(x, t)) := (-R_{11}^{-1} B_1' K_1(t)x, -R_{22}^{-1} B_2' K_2(t)x)$  is the feedback Nash equilibrium. The functions  $\psi_i$  are given by  $\psi_i(t) = K_i(t)x(t)$ .

*Proof (outline).* Suppose  $(\psi_1, \psi_2)$  in theorem 3.1 can be written as  $\psi_i = K_i x$ . Then the Nash equilibrium is given by  $\gamma_i^*(x, t) = -R_{ii}^{-1} B_i' \psi_i(t) = -R_{ii}^{-1} B_i' K_i x$ . Obviously  $(\gamma_1^*, \gamma_2^*) \in \Gamma_1^{fb} \times \Gamma_2^{fb}$ .  $\square$

*Remark.* When we allow for more general (e.g. nonlinear feedback) strategies, there exist many more Nash equilibria for the CLPS information structure.

In the previous section we have rewritten the Riccati equations (9–12) as a linear differential system (15), as was suggested in the papers [1, 2]. In that way it was easier to investigate the asymptotic properties of the open-loop Nash equilibrium and it also enabled us to calculate the solutions of (9–12). We will now see what happens if we try to rewrite the Riccati equations (23–26) for the feedback Nash equilibrium as a linear system, following the same approach as in the previous section. For the feedback Nash equilibrium the functions  $(\psi_1, \psi_2)$ , as described by theorem 3.1, satisfy the following differential equations:

$$(27) \quad \dot{\psi}_1 = -Q_1 x - (A' - K_2 S_2) \psi_1 - K_2 S_{02} \psi_2,$$

$$(28) \quad \dot{\psi}_2 = -Q_2 x - K_1 S_{01} \psi_1 - (A' - K_1 S_1) \psi_2.$$

This gives for the matrix  $M$

$$(29) \quad M = M(K_1, K_2) = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A' - K_2 S_2 & K_2 S_{02} \\ Q_2 & K_1 S_{01} & A' - K_1 S_1 \end{pmatrix}.$$

In the (realistic) case  $R_{12} = 0$ ,  $R_{21} = 0$ , (29) simplifies to

$$(30) \quad M = M(K_1, K_2) = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A' - K_2 S_2 & 0 \\ Q_2 & 0 & A' - K_1 S_1 \end{pmatrix}.$$

Note that, even in this special case,  $M$  depends on  $(K_1, K_2)$ , so that the resulting equations are still nonlinear. In the rest of this section we will study in detail the quadratic system of Riccati differential equations for the feedback Nash equilibrium in the most simple case where all parameters are scalar and  $R_{12} = 0$ ,  $R_{21} = 0$ . This analysis will show that the situation for the feedback Nash equilibrium is much more complicated than in the open-loop case.

**3.2. The scalar case.** Below we restrict our attention to the case in which all the system parameters are scalar<sup>1</sup>. Furthermore, we shall confine ourselves to the case where  $q_1, q_2, s_1$  and  $s_2$  are all strictly positive. If we rewrite the terminal value problem for  $(k_1(t), k_2(t))$  as an initial value problem for  $(k_1(\tau), k_2(\tau))$ , we get

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<sup>1</sup>To emphasize the fact that all system parameters are scalar we put them in lower case, hence e.g.  $q_1$  instead of  $Q_1$ .

$$(31) \quad \dot{k}_1 = 2ak_1 + q_1 - s_1k_1^2 - 2s_2k_1k_2,$$

$$(32) \quad \dot{k}_2 = 2ak_2 + q_2 - s_2k_2^2 - 2s_1k_1k_2,$$

$$(33) \quad k_1(0) = k_{1f},$$

$$(34) \quad k_2(0) = k_{2f}.$$

Now define

$$(35) \quad \sigma_i := s_i q_i, \quad i = 1, 2,$$

$$(36) \quad \kappa_i := s_i k_i, \quad i = 1, 2.$$

Then we get the following system of quadratic differential equations:

$$(37) \quad \dot{\kappa}_1 = 2a\kappa_1 + \sigma_1 - \kappa_1^2 - 2\kappa_1\kappa_2,$$

$$(38) \quad \dot{\kappa}_2 = 2a\kappa_2 + \sigma_2 - \kappa_2^2 - 2\kappa_1\kappa_2.$$

The study of planar quadratic systems in general is a very complicated topic, as e.g. can be seen in the survey papers [6, 12]. For example the famous 16th Hilbert problem, to determine the maximal number of limit cycles,  $H_d$ , for  $d$ th degree polynomial planar systems, has not been solved yet even for quadratic systems ( $d = 2$ ). Hence, in general we can expect complicated dependence on the parameters for the quadratic system (37–38); for instance in [12], Reyn finds 101 topologically different global phase portraits for a 6-parameter family of quadratic systems. In the following subsections we address some of the characteristics of the quadratic system (37–38) that one typically is interested in.

**3.3. Periodic solutions.** The first question we address is the determination of the maximal number of limit cycles for the quadratic system (37–38). This leads to the question of existence of periodic solutions. We recall a famous criterion due to Bendixson (introduced in the paper [5], see also e.g. [10]):

**Theorem 3.3 (Bendixson).** *Let  $f \in C^1(E)$ , where  $E$  is a simply connected region in  $\mathbb{R}^2$ . If the divergence of the vector field  $f$ ,  $\nabla \cdot f$ , is not identically zero and does not change sign in  $E$ , then the planar system  $\dot{x} = f(x)$  has no periodic solution lying entirely in  $E$ .*

Recall that the divergence of a vector field in  $\mathbb{R}^2$  is given by the trace of the Jacobian matrix. The divergence of the quadratic system (37–38) is given by

$$(39) \quad \nabla \cdot f = 4a - 4\kappa_1 - 4\kappa_2.$$

Hence, the divergence equals zero on the line

$$(40) \quad a - \kappa_1 - \kappa_2 = 0.$$

From theorem 3.3 it follows that if there would exist a periodic solution of the quadratic system (37–38), this solution would have to cross the line (40) at least two times. However, on the line (40) we have

$$\dot{\kappa}_1 = \sigma_1 + \kappa_1^2 > 0,$$

$$\dot{\kappa}_2 = \sigma_2 + \kappa_2^2 > 0,$$

and hence any solution of (37–38) can cross the line (40) at most once. We conclude therefore that there does not exist any periodic solution to the quadratic system (37–38), and thus there are no limit cycles.

**3.4. Critical points.** The question of determining critical points of the quadratic system (37–38) is closely related to the question of the existence of stationary feedback Nash equilibria. The critical points of the differential equations (37–38) are the intersection points of the hyperbolas, given by

$$(41) \quad 2a\kappa_1 + \sigma_1 - \kappa_1^2 - 2\kappa_1\kappa_2 = 0,$$

$$(42) \quad 2a\kappa_2 + \sigma_2 - \kappa_2^2 - 2\kappa_1\kappa_2 = 0.$$

Simple calculations show that hyperbola (41) has the asymptotes  $\kappa_1 = 0$  and  $\kappa_2 = a - \frac{1}{2}\kappa_1$  and hyperbola (42) has the asymptotes  $\kappa_2 = 0$  and  $\kappa_2 = 2a - 2\kappa_1$ . Furthermore hyperbola (41) intersects the  $\kappa_1$ -axis in the points where  $\kappa_1 = a \pm \sqrt{a^2 + \sigma_1}$ , and hyperbola (42) intersects the  $\kappa_2$ -axis in the points where  $\kappa_2 = a \pm \sqrt{a^2 + \sigma_2}$ . We are now able to prove the following lemma:

**Lemma 3.4.** *The hyperbolas (41) and (42) can only intersect in the first or third quadrant of the  $(\kappa_1, \kappa_2)$  plane.*

*Proof.* Suppose (41) and (42) intersect in a point  $S = (\bar{\kappa}_1, \bar{\kappa}_2)$  where  $\bar{\kappa}_1 > 0$ . Hyperbola (42) intersects the  $\kappa_2$ -axis in the points  $\kappa_2 = a \pm \sqrt{a^2 + \sigma_2}$ . Because the  $\kappa_1$ -axis is an asymptote of (42), in  $S$  we have either  $\bar{\kappa}_2 > 0$  or  $\bar{\kappa}_2 < 0$  and then  $\bar{\kappa}_2 < a - \sqrt{a^2 + \sigma_2}$ , i.e.  $S$  might be located either on the upper curve or on the lower curve of (42). Suppose, in the intersection point  $S$ ,  $\bar{\kappa}_2 < a - \sqrt{a^2 + \sigma_2}$ , i.e.  $S$  is located on the lower curve of (42). Elementary calculus shows that on (41) for  $\kappa_1 > 0$ ,  $\kappa_2 < a - \sqrt{a^2 + \sigma_2}$  iff  $\kappa_1 > \sqrt{a^2 + \sigma_2} + \sqrt{a^2 + \sigma_1} + \sigma_2$ . Hence, necessarily  $\bar{\kappa}_1 > \sqrt{a^2 + \sigma_2} + \sqrt{a^2 + \sigma_1} + \sigma_2$ . Moreover, because the intersection point  $S$  lies on (41) to the right of the  $\kappa_2$ -axis, we know  $S$  has to be located above the asymptote  $\kappa_2 = a - \frac{1}{2}\kappa_1$ , hence



$\bar{\kappa}_2 > a - \frac{1}{2}\bar{\kappa}_1$ . Similarly,  $S$  has to be located on (42) below the asymptote  $\kappa_2 = 2a - 2\kappa_1$ . Hence  $\bar{\kappa}_1 < \frac{2a}{3}$ . But this contradicts the fact that  $\bar{\kappa}_1 > \sqrt{a^2 + \sigma_2} + \sqrt{a^2 + \sigma_1 + \sigma_2}$ . Therefore  $\bar{\kappa}_2 > 0$ , and thus there exists no intersection point in the fourth quadrant. Along the same lines one can prove that there exists no intersection point in the second quadrant.  $\square$

By taking a closer look at the proof of the previous lemma we can identify two square regions in  $\mathbb{R}^2$  in which possible critical points can be located. Define the following two regions in  $\mathbb{R}^2$ ,

$$(43) \quad G_1 := \left(0, a + \sqrt{a^2 + \sigma_1}\right) \times \left(0, a + \sqrt{a^2 + \sigma_2}\right),$$

$$(44) \quad G_2 := \left(a - \sqrt{a^2 + \sigma_1}, 0\right) \times \left(a - \sqrt{a^2 + \sigma_2}, 0\right).$$

We have the following lemma:

**Lemma 3.5.** *The critical points of the quadratic system (37-38) are located in the regions  $G_1$  and  $G_2$ . Moreover, each of the regions contains at least one critical point.*

*Proof.* The region  $G_1$  lies entirely in the first quadrant. (41) intersects the  $\kappa_1$ -axis, in the point where  $\kappa_1 = a + \sqrt{a^2 + \sigma_1}$ , and hence any critical point in the first quadrant has to be located to the left of  $\kappa_1 = a + \sqrt{a^2 + \sigma_1}$ . Similarly, any critical point in the first quadrant has to be located below the line  $\kappa_2 = a + \sqrt{a^2 + \sigma_2}$ . Hence, any critical point in the first quadrant has to be located in  $G_1$ , and similarly any critical point in the third quadrant has to be located in  $G_2$ . Furthermore, it is easily seen that (41) enters  $G_1$  in the point  $(0, a + \sqrt{a^2 + \sigma_2})$ , and leaves  $G_1$  through the line  $\kappa_1 = a + \sqrt{a^2 + \sigma_1}$ . Hyperbola (42) enters  $G_1$  through the line  $\kappa_2 = a + \sqrt{a^2 + \sigma_2}$ , and leaves  $G_1$  in the point  $(a + \sqrt{a^2 + \sigma_1}, 0)$ . Necessarily, (41) and (42) have to intersect at least once in  $G_1$  (and similarly at least once in  $G_2$ ).  $\square$

**Lemma 3.6.** *In every critical point  $S = (\bar{\kappa}_1, \bar{\kappa}_2)$  located in  $G_1$  there holds  $a - \bar{\kappa}_1 - \bar{\kappa}_2 < 0$ , and in every critical point  $T = (\bar{\kappa}_1, \bar{\kappa}_2)$  located in  $G_2$  there holds  $a - \bar{\kappa}_1 - \bar{\kappa}_2 > 0$ .*

*Proof.* Let  $S = (\bar{\kappa}_1, \bar{\kappa}_2)$  be a critical point in  $G_1$ . Because  $S$  is located on the hyperbola (41), we know  $S$  is located above the asymptote  $\kappa_2 = a - \frac{1}{2}\kappa_1$ , and thus

$$\bar{\kappa}_2 > a - \frac{1}{2}\bar{\kappa}_1.$$

And hence

$$a - \bar{\kappa}_1 - \bar{\kappa}_2 < a - \bar{\kappa}_1 - a + \frac{1}{2}\bar{\kappa}_1 = -\frac{1}{2}\bar{\kappa}_1 < 0.$$

The proof that for every critical point  $T = (\bar{\kappa}_1, \bar{\kappa}_2)$  located in  $G_2$  there holds  $a - \bar{\kappa}_1 - \bar{\kappa}_2 > 0$ , goes along the same lines.  $\square$

*Remark.* The property  $a - \bar{\kappa}_1 - \bar{\kappa}_2 < 0$  can be interpreted in terms of closed-loop stability of the associated stationary linear feedback strategies  $\tilde{\gamma}_i = -\frac{b_i}{r_{ii}}\bar{k}_i x$ .

We see that the system (37–38) has at least two critical points. Because the system is quadratic we also know that the system (37–38) has at most four critical points. Furthermore, the system (37–38) can only have critical points of multiplicity up to 3, because of the location of the critical points in the areas  $G_1$  and  $G_2$ .

**Lemma 3.7.** *If the quadratic system (37–38) has a critical point of multiplicity 2 or 3, then the system parameters have to satisfy the equation*

$$(45) \quad a^8 + (6\sigma_1\sigma_2 - 6\sigma_1^2 - 6\sigma_2^2)a^4 + (12\sigma_1^2\sigma_2 + 12\sigma_1\sigma_2^2 - 8\sigma_1^3 - 8\sigma_2^3)a^2 - 9\sigma_1^2\sigma_2^2 + 6\sigma_1^3\sigma_2 + 6\sigma_1\sigma_2^3 - 3\sigma_1^4 - 3\sigma_2^4 = 0$$

*Proof.* Suppose  $S$  is a critical point of higher multiplicity of the system (37–38). Then the tangent of (41) and the tangent of (42) in  $S$  have to coincide. Note that, because of the fact that on the line  $a - \kappa_1 - \kappa_2 = 0$ , both  $\dot{\kappa}_1 > 0$  and  $\dot{\kappa}_2 > 0$ , in any critical point  $a - \kappa_1 - \kappa_2 \neq 0$ . We find that in  $S$  necessarily

$$(46) \quad \begin{cases} 2a\kappa_1 + \sigma_1 - \kappa_1^2 - 2\kappa_1\kappa_2 = 0 \\ 2a\kappa_2 + \sigma_2 - \kappa_2^2 - 2\kappa_1\kappa_2 = 0 \\ (a - \kappa_1 - \kappa_2)^2 - \kappa_1\kappa_2 = 0 \end{cases}$$

Using a Gröbner basis (calculated with Maple V) for the system of equations (46),  $\kappa_1$  and  $\kappa_2$  are eliminated from the equations (46), and so we find that the system parameters have to satisfy the equation (45).  $\square$

From the previous lemma, we see that bifurcations can occur where the system parameters satisfy equation (45). When equation (45) is not satisfied, all critical points will have multiplicity 1. In the point where equation (45) is satisfied two (or three) critical points may coincide. In the case all critical points have multiplicity 1, there will be either two or four critical points. There are three possibilities:

- (i) The system (37–38) has exactly two critical points, one of them lies in  $G_1$ , the other in  $G_2$ . (see figure 1)
- (ii) The system has four different critical points, one of them lies in  $G_1$  and all the others in  $G_2$ . (see figure 2)
- (iii) The system has four different critical points, one of them lies in  $G_2$  and all the others in  $G_1$ . (see figure 3)

We will illustrate the above results in an example.

**Example 3.8.** In this example we take  $\sigma_1 = 0.25$  and  $\sigma_2 = 0.2$ . Then (37–38) is given by

$$\begin{aligned} \dot{\kappa}_1 &= 2a\kappa_1 + 0.25 - \kappa_1^2 - 2\kappa_1\kappa_2, \\ \dot{\kappa}_2 &= 2a\kappa_2 + 0.2 - \kappa_2^2 - 2\kappa_1\kappa_2. \end{aligned}$$

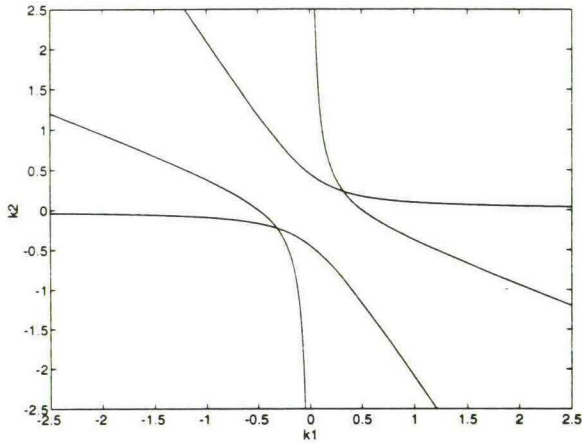
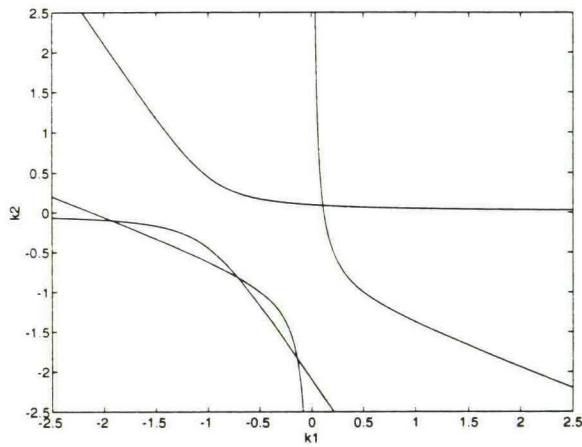


FIGURE 1. Case (i), two critical points

FIGURE 2. Case (ii), four critical points, one in  $G_1$ .

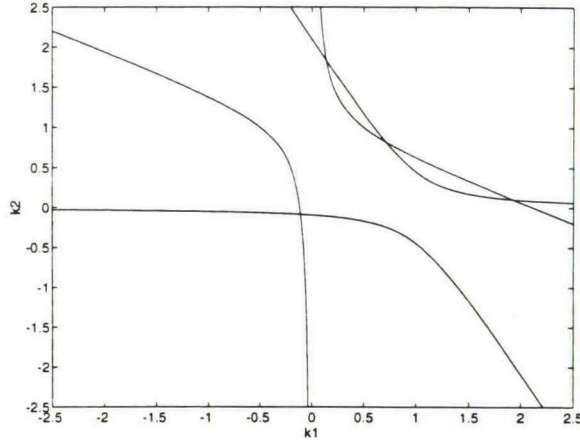


FIGURE 3. Case 3: four critical points, three in  $G_1$ .

The equation (45) is given by

$$16000a^8 - 50400a^4 + 12960a^2 - 1323 = 0,$$

which has the (real) solutions  $a \approx \pm 0.6383$ . First we study the case  $a = 1$ . In this case the critical points are

$$\begin{aligned} P_1 &\approx (-0.110, -0.087), \\ P_2 &\approx (1.925, 0.102), \\ P_3 &\approx (0.712, 0.820), \\ P_4 &\approx (0.139, 1.832). \end{aligned}$$

Now the case  $a = 0$ . Then the critical points are

$$\begin{aligned} P_1 &\approx (-0.321, -0.230), \\ P_2 &\approx (0.321, 0.230). \end{aligned}$$

Finally, we study the case  $a = -1$ . The critical points are:

$$\begin{aligned} P_1 &\approx (-0.139, -1.832), \\ P_2 &\approx (-0.712, -0.819), \\ P_3 &\approx (-1.925, -0.102), \\ P_4 &\approx (0.110, 0.087). \end{aligned}$$



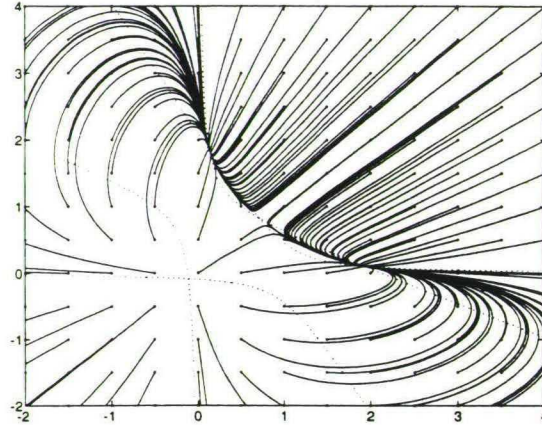


FIGURE 4. Solutions of the differential equations for  $a = 1$ ,  $\sigma_1 = 0.25$  and  $\sigma_2 = 0.2$ .

In the bifurcation at  $a \approx 0.6383$ , the system changes from having four critical points (for  $a > 0.6383$ ) towards a situation in which there are two critical points (for  $a < 0.6383$ ). In the bifurcation at  $a \approx -0.6383$  the system changes again from two critical points (for  $a > -0.6383$ ) to four critical points (for  $a < -0.6383$ ).

For the case  $a = 1$  we have calculated some solutions of the differential equations (37–38). (see figure 4)

**3.5. The behaviour at infinity.** Finally we analyze the critical points at infinity. We study the behaviour of trajectories “at infinity” by studying the flow of the quadratic system (37–38) on the so-called Poincaré sphere. This approach was introduced by Poincaré in the paper [11]. A description of this theory can be found in [10, pp. 248–269]. When we consider a flow of a dynamical system on  $\mathbb{R}^2$ , given by

$$(47) \quad \begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases}$$

where  $P$  and  $Q$  are polynomial functions of  $x$  and  $y$ , then the critical points at infinity for the polynomial system (47) occur at the points  $(X, Y, 0)$  on the equator of the Poincaré sphere where  $X^2 + Y^2 = 1$  and

$$(48) \quad XQ_m(X, Y) - YP_m(X, Y) = 0.$$

Here,  $m$  is the maximal degree of the terms in  $P$  and  $Q$ ,  $P_m$  and  $Q_m$  denote the polynomials consisting of the terms of degree  $m$ . The solutions  $X, Y$  of (48), with  $X^2 + Y^2 = 1$ , can be

found at the polar angles  $\theta_j$  and  $\theta_j + \pi$  satisfying

$$(49) \quad G_{m+1}(\theta) \equiv (\cos \theta, \sin \theta) \cos \theta Q_m - P_m(\cos \theta, \sin \theta) \sin \theta = 0.$$

For the system (37–38),  $m = 2$  and the polynomials  $P$  and  $Q$  are given by

$$(50) \quad P(x, y) = 2ax + \sigma_1 - x^2 - 2xy,$$

$$(51) \quad Q(x, y) = 2ay + \sigma_2 - y^2 - 2xy.$$

Thus,  $P_2$  and  $Q_2$  become

$$(52) \quad P_2(x, y) = -x^2 - 2xy,$$

$$(53) \quad Q_2(x, y) = -y^2 - 2xy.$$

The critical points at infinity for the system (37–38) can now be found by solving the following equations:

$$(54) \quad XQ_2(X, Y) - YP_2(X, Y) = 0,$$

$$(55) \quad X^2 + Y^2 = 1,$$

which is equivalent to

$$(56) \quad XY^2 - YX^2 = 0,$$

$$(57) \quad X^2 + Y^2 = 1.$$

We find the following points:

	$X$	$Y$	$\theta$	nature
$P_1$	1	0	0	saddle
$P_2$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{4}\pi$	unstable node
$P_3$	0	1	$\frac{1}{2}\pi$	saddle
$P_4$	-1	0	$\pi$	saddle
$P_5$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$\frac{5}{4}\pi$	stable node
$P_6$	0	-1	$\frac{3}{2}\pi$	saddle

Note that the behaviour at infinity of the system (37–38) is independent of the parameters  $a, \sigma_1, \sigma_2$ .

**3.6. The nature of the critical points.** In this subsection we study the nature of the finite critical points of the quadratic system (37–38). The Jacobian of the system (37–38) is given by

$$(58) \quad Df(\kappa_1, \kappa_2) = \begin{pmatrix} 2(a - \kappa_1 - \kappa_2) & -2\kappa_1 \\ -2\kappa_2 & 2(a - \kappa_1 - \kappa_2) \end{pmatrix}$$

The eigenvalues of  $Df(\kappa_1, \kappa_2)$ , for  $(\kappa_1, \kappa_2)$  in  $G_1$  or  $G_2$ , are given by

$$(59) \quad \lambda_{1,2} = 2(a - \kappa_1 - \kappa_2) \pm 2\sqrt{\kappa_1\kappa_2}.$$

We already noted (see proof of lemma 3.7), that in any critical point of multiplicity 1,  $(a - \kappa_1 - \kappa_2)^2 \neq \kappa_1\kappa_2$ , hence any critical point of multiplicity 1 is hyperbolic. Moreover, since  $\lambda_1$  and  $\lambda_2$  are both real, all hyperbolic critical points are either nodes or saddles, there are no foci.

On the projective plane (the projection of the upper hemisphere of the Poincaré sphere onto the unit disk), when we identify the antipodal points on the unit circle, we know for the vectorfield, defined by (37–38), by the Poincaré Index theorem, that  $n - s = 1$ , where  $n$  is the number of nodes and  $s$  is the number of saddles. In the previous section we determined the nature of the critical points at infinity (2 saddles and 1 node), hence

$$(60) \quad n_f - s_f = 1 - n_\infty + s_\infty = 2,$$

where  $n_f, s_f$  are the number of "finite nodes" and "finite saddles", respectively, and  $n_\infty, s_\infty$  are the number of nodes and saddles respectively at infinity.

Case (i): two finite critical points of multiplicity 1. In the case there are exactly two finite critical points, we deduce from (60) that these points necessarily have to be nodes. Because of the fact that for the critical point in  $G_1$ , by lemma 3.6,  $a - \kappa_1 - \kappa_2 < 0$ , we know that in this point, in agreement with (59), the eigenvalues of the Jacobian  $\lambda_1$  and  $\lambda_2$  are both negative. Hence the critical point in  $G_1$  is a stable node. Similarly, the critical point in  $G_2$  is an unstable node.

Case (ii): four finite critical points, one of them in  $G_1$ . We have four finite critical points, hence  $n_f + s_f = 4$ . Moreover, by (60) we know  $n_f - s_f = 2$ . Hence,  $n_f = 3$  and  $s_f = 1$ : there are three nodes and one saddle. Denote the critical point in  $G_1$  by  $(\bar{\kappa}_1, \bar{\kappa}_2)$ . We will show that this point is a stable node. We can (locally) interpret hyperbola (41) as a function, given by  $\kappa_2 = h_1(\kappa_1)$ , and similarly we can (locally) interpret hyperbola (42) as a function given by  $\kappa_2 = h_2(\kappa_1)$ . Then, the derivative of  $h_1$  in a point  $(\kappa_1, \kappa_2)$  on hyperbola (41), is given by

$$(61) \quad h'_1(\kappa_1, \kappa_2) = \frac{a - \kappa_1 - \kappa_2}{\kappa_1},$$

and the derivative of  $h_2$  in a point  $(\kappa_1, \kappa_2)$  on hyperbola (42), is given by

$$(62) \quad h'_2(\kappa_1, \kappa_2) = \frac{\kappa_2}{a - \kappa_1 - \kappa_2}.$$

Furthermore, the determinant of the Jacobian (58) is given by

$$(63) \quad \det Df(\kappa_1, \kappa_2) = 4 \left( (a - \kappa_1 - \kappa_2)^2 - \kappa_1\kappa_2 \right).$$

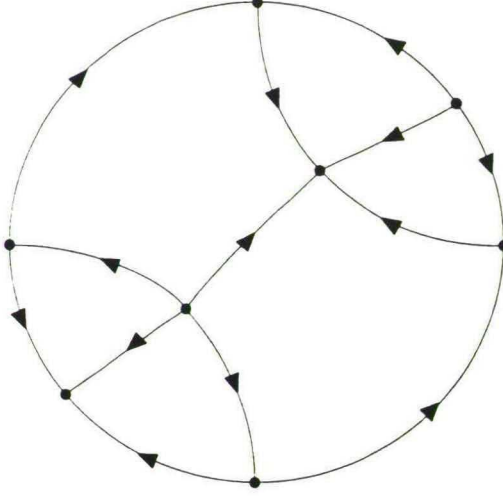


FIGURE 5. Case (i), two finite critical points.

Since, in  $G_1$ ,  $\kappa_1 > 0$ ,  $\kappa_2 > 0$  and  $(a - \kappa_1 - \kappa_2) < 0$  we find

$$(64) \quad \det Df(\kappa_1, \kappa_2) > 0 \Leftrightarrow h'(\kappa_1, \kappa_2) < 0,$$

where

$$(65) \quad h(\kappa_1) = h_1(\kappa_1) - h_2(\kappa_1).$$

Now it is easily verified, that the critical point in  $G_1$  is the point where  $h(\bar{\kappa}_1) = 0$ , and moreover that  $h(\kappa_1)$  changes sign from positive to negative when  $\kappa_1$  is increased. Hence,

$$(66) \quad h'(\bar{\kappa}_1, \bar{\kappa}_2) < 0,$$

and thus

$$(67) \quad \det Df(\bar{\kappa}_1, \bar{\kappa}_2) > 0,$$

meaning that the eigenvalues  $\lambda_1$  and  $\lambda_2$  are both negative. Thus,  $(\bar{\kappa}_1, \bar{\kappa}_2)$  is a stable node.

Case (iii): four finite critical points, one of them in  $G_2$ . Similarly as in case (ii), we find  $n_f = 3$  and  $s_f = 1$ . The only critical point in  $G_2$  is an unstable node, the remaining three critical points in  $G_1$  now consist of two stable nodes and one saddle.

For the three cases, with finite critical points of multiplicity 1, we find the global phase portraits, as sketched in figures 5,6 and 7.



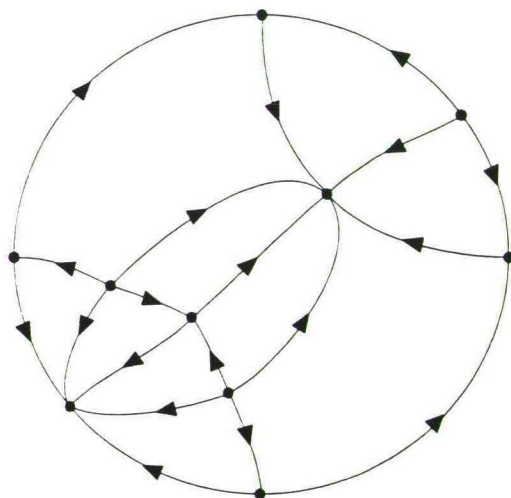


FIGURE 6. Case (ii), four finite critical points, one in  $G_1$ .

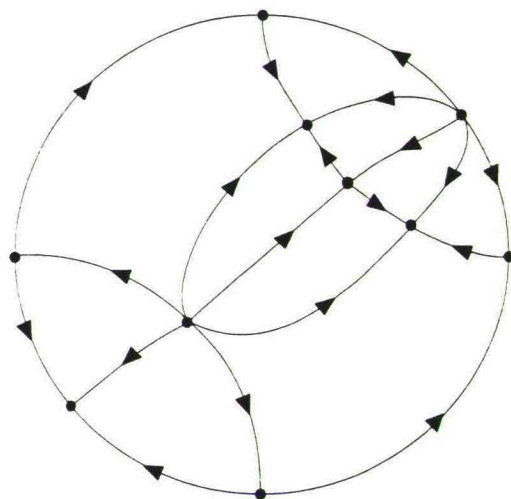


FIGURE 7. Case (iii), four finite critical points, three in  $G_1$ .

Apart from the bifurcations, we see that there exist three topologically different phase portraits of the system (37–38). Even more possibilities can be expected when the dimension of the state space or the number of players is increased. The analysis in this section can not straightforwardly be generalised to more dimensions or to the general  $N$ -player case. In our opinion it is a rather complicated task to perform a more general multidimensional,  $N$ -player, analysis. However, since in the 1-player case (the LQ optimal control problem) we know that the behaviour in the multivariable case is similar to the behaviour in the scalar case, we believe that our analysis provides some clues to what can be expected in the more general case.

The most important observation we have made, is that it is possible that there exist several different stable critical points for the coupled system of Riccati differential equations (23–26). In that case, even over a longer (finite) horizon, the solutions of the coupled system of Riccati differential equations depend heavily on the terminal conditions  $(K_{1f}, K_{2f})$ .

#### 4. THE INFINITE HORIZON FEEDBACK NASH EQUILIBRIUM.

In this section we consider the feedback Nash equilibrium in stationary strategies for the differential game over an infinite time horizon. By a stationary strategy, we mean a linear time-invariant feedback strategy. We study the following cost functionals:

$$(68) \quad \mathcal{L}_1(u_1, u_2) = \int_0^\infty \{x(t)'Q_1x(t) + u_1(t)'R_{11}u_1(t)\} dt,$$

$$(69) \quad \mathcal{L}_2(u_1, u_2) = \int_0^\infty \{x(t)'Q_2x(t) + u_2(t)'R_{22}u_2(t)\} dt,$$

with  $Q_i \geq 0$  and  $R_{ii} > 0$ .

We find the following lemma:

**Lemma 4.1.** *Suppose  $(C_i, D_i)$  are such that  $C_i' C_i = Q_i$ ,  $D_i' C_i = 0$  and  $D_i' D_i = R_{ii}$ . Suppose that there exist  $(K_1, K_2)$  satisfying the coupled algebraic Riccati equations*

$$(70) \quad A'K_1 + K_1A + Q_1 - K_1S_1K_1 - K_1S_2K_2 - K_2S_2K_1 = 0,$$

$$(71) \quad A'K_2 + K_2A + Q_2 - K_2S_2K_2 - K_2S_1K_1 - K_1S_1K_2 = 0,$$

such that  $K_1$  is the smallest real positive semidefinite solution of (70) for given  $K_2$  and  $K_2$  is the smallest real positive semidefinite solution of (71) for given  $K_1$ , and moreover  $K_1$  and  $K_2$  are such that the systems  $(A - B_2R_{22}^{-1}B_2'K_2, B_1, C_1, D_1)$  and  $(A - B_1R_{11}^{-1}B_1'K_1, B_2, C_2, D_2)$  are both output stabilizable. Then the strategies  $\gamma_i$ , given by  $u_i = \gamma_i(x) = -R_{ii}^{-1}B_i'K_ix$ , are a feedback Nash equilibrium in stationary strategies.

*Proof.* Suppose the second player plays some stationary feedback strategy  $\gamma_2(x) = F_2x$ , where  $F_2$  is such that the system  $(A + B_2F_2, B_1, C_1, D_1)$  is output stabilizable. To obtain the best response for player 1, player 1 has to solve the linear-quadratic optimal control problem

$$\min_{u_1} \int_0^\infty \{x(t)'Q_1x(t) + u_1(t)'R_{11}u_1(t)\} dt,$$

subject to

$$\dot{x} = (A + B_2F_2)x + B_1u_1, x(0) = x_0.$$

Because  $(A + B_2F_2, B_1, C_1, D_1)$  is output stabilizable, the optimal  $u_1$  is given by the stationary linear feedback strategy  $u_1 = \gamma_1(x) = -R_{11}^{-1}B_1'Px$ , where  $P$  is the smallest real positive semidefinite solution of the algebraic Riccati equation, given by

$$(A + B_2F_2)'P + P(A + B_2F_2) - PS_1P + Q_1 = 0.$$

Now suppose player 2 plays the strategy  $\gamma_2(x) = -R_{22}^{-1}B_2'K_2x$ , for some  $K_2$ , such that the system  $(A - B_2R_{22}^{-1}B_2', B_1, C_1, D_1)$  is output stabilizable. Then, the best response against this strategy for player 1 is to play the strategy  $\gamma_1(x) = -R_{11}^{-1}B_1'K_1x$ , where  $K_1$  is the smallest real positive semidefinite solution of (70) for given  $K_2$ . Similarly, for given  $K_1$  such that the system  $(A - B_1R_{11}^{-1}B_1'K_1, B_2, C_2, D_2)$  is output stabilizable, the best response of player 2 against  $\gamma_1(x) = -R_{11}^{-1}B_1'K_1x$  is to play  $\gamma_2(x) = -R_{22}^{-1}B_2'K_2x$ , where  $K_2$  is the smallest real positive semidefinite solution of (71) for given  $K_1$ . Hence,  $(\gamma_1(x), \gamma_2(x)) = (-R_{11}^{-1}B_1'K_1x, -R_{22}^{-1}B_2'K_2x)$  is a Nash equilibrium in stationary linear feedback strategies.  $\square$

*Remark.* Note that, although we require the smallest real symmetric solutions of the coupled Riccati equations, we do not necessarily have uniqueness by this lemma.

*Remark.* Taking a closer look at the proof of this lemma, we see that the best response against any stationary linear feedback strategy, is again a stationary linear feedback strategy.

In the scalar case analyzed in the previous section, all (dynamic) equilibria in the first quadrant are also stationary feedback Nash equilibria. This illustrates the possible nonuniqueness of stationary feedback Nash equilibria. We also see that the criterion of dynamic stability does distinguish between these equilibria, but only partly: the nonuniqueness is reduced, but not eliminated completely.

## 5. CONCLUSIONS.

In this paper we have studied the asymptotic properties of some different Nash equilibria in two-player, nonzero-sum, linear-quadratic differential games. For the open-loop case we have seen in section two, that there can exist at most one stable (in the dynamic sense) stationary Nash equilibrium. We have also seen that there are examples, in which there exists no stable stationary Nash equilibrium at all. In section three we have seen that the situation for the feedback Nash equilibrium is more complicated. We found out that it is possible that there exist several different stable Nash equilibria. Moreover, we saw that the asymptotic behaviour of a feedback Nash equilibrium depends heavily on the specified terminal conditions. Finally,

in section four, we have studied stationary feedback Nash equilibria for games over an infinite time horizon. We saw that, although in the finite horizon case there is an unique feedback Nash equilibrium, in the infinite horizon case there may be nonuniqueness problems.

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